Problemă propusă pentru gazeta matematică tipărită (clasa a IX-a)

Rezultatul nu exista în O.E.I.S. până la data de 08.04.2022

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Fie un numar natural $n = \overline{a_1 a_2 \dots a_k}$, *unde* $a_1, a_2, \dots, a_k \in \{0,1,2, \dots, 9\}$, $a_1 \neq 0$. *Considerand* $P(n)$ *produsul cifrelor numarului* n, *iar* $S(n)$ *suma produselor cifrelor numerelor pana la numarul* n:

$S(n) = P(1) + P(2) + \cdots + P(n)$. *Sa se arate ca*:

$$S(n) = \left[\frac{a_1(a_1-1)}{2} + 1\right] 45^{k-1} + \left[a_1 \cdot \frac{a_2(a_2-1)}{2} + 1\right] 45^{k-2} +$$

$$+ \left[a_1 a_2 \cdot \frac{a_3(a_3-1)}{2} + 1\right] 45^{k-3} + \dots + \left[a_1 a_2 \dots a_{k-2} \cdot \frac{a_{k-1}(a_{k-1}-1)}{2} + 1\right] 45^1$$

$$+ a_1 a_2 \dots a_{k-1} \cdot \frac{a_k(a_k-1)}{2} + a_1 a_2 \dots a_k .$$

Prof. Mihai Teodor, Focșani

P.S.1. *Relatia de mai sus am demonstrat – o inductiv, cu ajutorul unei formule de recurenta.*

P.S.2. *De asemenea, relatia de mai sus am demonstrat – o si cu ajutorul unui program* C^{++}.

O alta forma echivalenta pentru $S(n)$ *este*:

$$S(\overline{a_1 a_2 \dots a_k}) = B_1 \cdot 45^{k-1} + B_2 \cdot 45^{k-2} + \cdots + B_{k-1} \cdot 45 + B_k.$$

$unde\ B_i = \left(\prod_{j=1}^{i} a_j\right) \cdot \frac{a_i - 1}{2} + 1\ , (\forall) i = 1, k-1, iar\ B_k = a_1 a_2 \dots a_k \frac{a_k + 1}{2}$

$care\ se\ mai\ scrie\ si\ B_k = \left(a_1 a_2 \dots a_k \frac{a_k - 1}{2} + a_1 a_2 \dots a_k\right) \cdot 45^0.$

A treia forma echivalenta , asa cum apare in ***GMB2*** */2022 si* ***O.E.I.S.*** *este*

Fie un numar natural $n = \overline{a_1 a_2 \dots a_k}, unde\ a_1, a_2, \dots, a_k \in \{0,1,2, \dots, 9\},$

$a_1 \neq 0.$ *Atunci suma produselor cifrelor numerelor pana la* "n" *este*

$$S(\overline{a_1 a_2 \dots a_k}) = a_1 a_2 \dots a_k + \frac{45^k - 45}{44} + \frac{1}{2} \sum_{i=1}^{k} 45^{k-i}\ a_1 a_2 \dots a_i (a_i - 1)$$

Solutie: *Sa stabilim mai intai doua relatii*: $(*)$ *si* $(**)$.

$(*)\ S(\overline{a_1 a_2 \dots a_{k+1}}) = S(\overline{a_1 a_2 \dots a_k 0}) + a_1 a_2 \dots a_k \cdot S(a_{k+1})$

Demonstratie:

$S(\overline{a_1 a_2 \dots a_k a_{k+1}}) = P(1) + P(2) + \dots + P(\overline{a_1 a_2 \dots a_k 0}) + P(\overline{a_1 a_2 \dots a_k 1}) +$

$+P(\overline{a_1 a_2 \dots a_k 2}) + \dots + P(\overline{a_1 a_2 \dots a_k a_{k+1}}) = S(\overline{a_1 a_2 \dots a_k 0}) + a_1 a_2 \dots a_k \cdot 1 +$

$a_1 a_2 .. a_k 2 + \dots + a_1 a_2 \dots a_k a_{k+1} = S(\overline{a_1 a_2 \dots a_k 0}) + a_1 a_2 .. a_k (1 + 2 + \dots + a_{k+1})$

$= S(\overline{a_1 a_2 \dots a_k 0}) + a_1 a_2 \dots a_k \cdot S(a_{k+1})\ \ (\boldsymbol{A})$

$(**):\ S(\overline{a_1 a_2 \dots a_k 0}) = 45 + 45 \cdot S(\overline{a_1 a_2 \dots a_k}) - 45 \cdot a_1 \cdot a_2 \cdot \dots \cdot a_k.$

Demonstratie:

$S(\overline{a_1 a_2 \dots a_k 0}) = P(1) + P(2) + \dots + P(9) +$

$+P(10) + P(20) + \dots + P(\overline{a_1 a_2 \dots a_k 0}) +$

$+P(11) + P(21) + \dots + P(\overline{a_1 a_2 \dots a_k 1}) +$

$+. \ldots \ldots \ldots \ldots$

$+P(19) + P(29) + \cdots + P(\overline{a_1 a_2 \ldots a_k 9})$-

$- P(\overline{a_1 a_2 \ldots a_k 1}) - P(\overline{a_1 a_2 \ldots a_k 2}) - \ldots - P(\overline{a_1 a_2 \ldots a_k 9}) =$

$= P(1) + P(2) + \cdots + P(9) +$

$+0 \cdot P(1) + 0 \cdot P(2) + \cdots + 0 \cdot P(\overline{a_1 a_2 \ldots a_k}) +$

$+1 \cdot P(1) + 1 \cdot P(2) + \cdots + 1 \cdot P(\overline{a_1 a_2 \ldots a_k}) +$

$+. \ldots \ldots \ldots \ldots$

$+9 \cdot P(1) + 9 \cdot P(2) + \cdots + 9 \cdot P(\overline{a_1 a_2 \ldots a_k}) -$

$-1 \cdot P(\overline{a_1 a_2 \ldots a_k}) - 2 \cdot P(\overline{a_1 a_2 \ldots a_k}) - \ldots - 9 \cdot P(\overline{a_1 a_2 \ldots a_k}) =$

$= 45 + 0 \cdot S(\overline{a_1 a_2 \ldots a_k}) + 1 \cdot S(\overline{a_1 a_2 \ldots a_k}) + \cdots + 9 \cdot S(\overline{a_1 a_2 \ldots a_k})$

$-45 \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_k. \ (\boldsymbol{A})$

$= 45 + 45 \cdot S(\overline{a_1 a_2 \ldots a_k}) - 45 \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_k \ .$

***Demonstratia problemei propuse*(*pentru forma* 1):**

(*Metoda* 1)

$k = 1 \Rightarrow S(a_1) = P(1) + P(2) + \cdots + P(a_1) = \left[\frac{a_1(a_1+1)}{2}\right] =$

$= \frac{a_1(a_1-1)}{2} + a_1 \ (\boldsymbol{A})$

$k = 2 \Rightarrow S(a_1 a_2) = \left[\frac{a_1(a_1-1)}{2} + 1\right] \cdot 45 + a_1 \cdot \frac{a_2(a_2-1)}{2} + a_1 a_2 \ . \ (\boldsymbol{A})$

$P(k) \longrightarrow P(k+1)$

$P(k+1): S(\overline{a_1 a_2 \ldots a_k a_{k+1}}) =$

$= \left[\frac{a_1(a_1-1)}{2} + 1\right] \cdot 45^k + \left[a_1 \cdot \frac{a_2(a_2-1)}{2} + 1\right] \cdot 45^{k-1} +$

$$+\cdots+\left[a_1a_2\ldots a_{k-1}\cdot\frac{a_k(a_k-1)}{2}+1\right]\cdot 45+a_1a_2\ldots a_k\cdot\frac{a_{k+1}(a_{k+1}-1)}{2}$$

$+a_1a_2\ldots a_ka_{k+1}$; *Membrul drept se mai scrie astfel*:

$45\cdot S(\overline{a_1a_2\ldots a_k})-45\cdot a_1a_2\ldots a_k+45+\ a_1a_2\ldots a_k\cdot\frac{a_{k+1}(a_{k+1}-1)}{2}+$

$+a_1a_2\ldots a_ka_{k+1}\overset{(**)}{=}S(\overline{a_1a_2\ldots a_k0})+a_1a_2\ldots a_k\cdot\frac{a_{k+1}(a_{k+1}-1)}{2}+$

$+a_1a_2\ldots a_ka_{k+1}=S(\overline{a_1a_2\ldots a_k0})+a_1a_2\ldots a_k\cdot S(a_{k+1})\quad(\boldsymbol{A})conform\ (*)$

Deci $P(k)$ $(\boldsymbol{A})$ $(\forall)\ k\in\mathbb{N}^*$.

($\boldsymbol{Metoda\ 2}$) *direct, se poate demonstra inductiv relatia initiala folosind relatia de recurenta urmatoare*:

$S(\overline{a_1a_2\ldots a_k})=45\cdot S(\overline{a_1a_2\ldots a_{k-1}})-45\cdot a_1a_2\ldots a_{k-1}+45+$

$+a_1a_2\ldots a_k\frac{a_k+1}{2}$

Deci sa demonstram propozitia $P(k+1)\Rightarrow S(\overline{a_1a_2\ldots a_{k+1}})=$

$=45\cdot S(\overline{a_1a_2\ldots a_k})-45\cdot a_1a_2\ldots a_k+45+$ **P(k)**

$=a_1a_2\ldots a_{k+1}\cdot\frac{a_{k+1}+1}{2}=\left(conform\ propozitiei\ P(k)\right)=$

$=\left[\frac{a_1(a_1-1)}{2}+1\right]\cdot 45^k+\left[a_1\cdot\frac{a_2(a_2-1)}{2}+1\right]\cdot 45^{k-1}+\cdots+$

$+\left[\frac{a_1a_2\ldots a_k(a_k-1)}{2}+1\right]\cdot 45+a_1a_2\ldots a_k\cdot\frac{a_{k+1}(a_{k+1}+1)}{2}\Rightarrow$

$\Rightarrow P(k+1)\ este\ \boldsymbol{adevarata}.\ \ Deci\ P(n)\ (\boldsymbol{A})(\forall)k\in\mathbb{N}^*.$

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Observatie *Rezultatul nu exista in* ***O.E.I.S.pana la data de*** **08.04 2022.**

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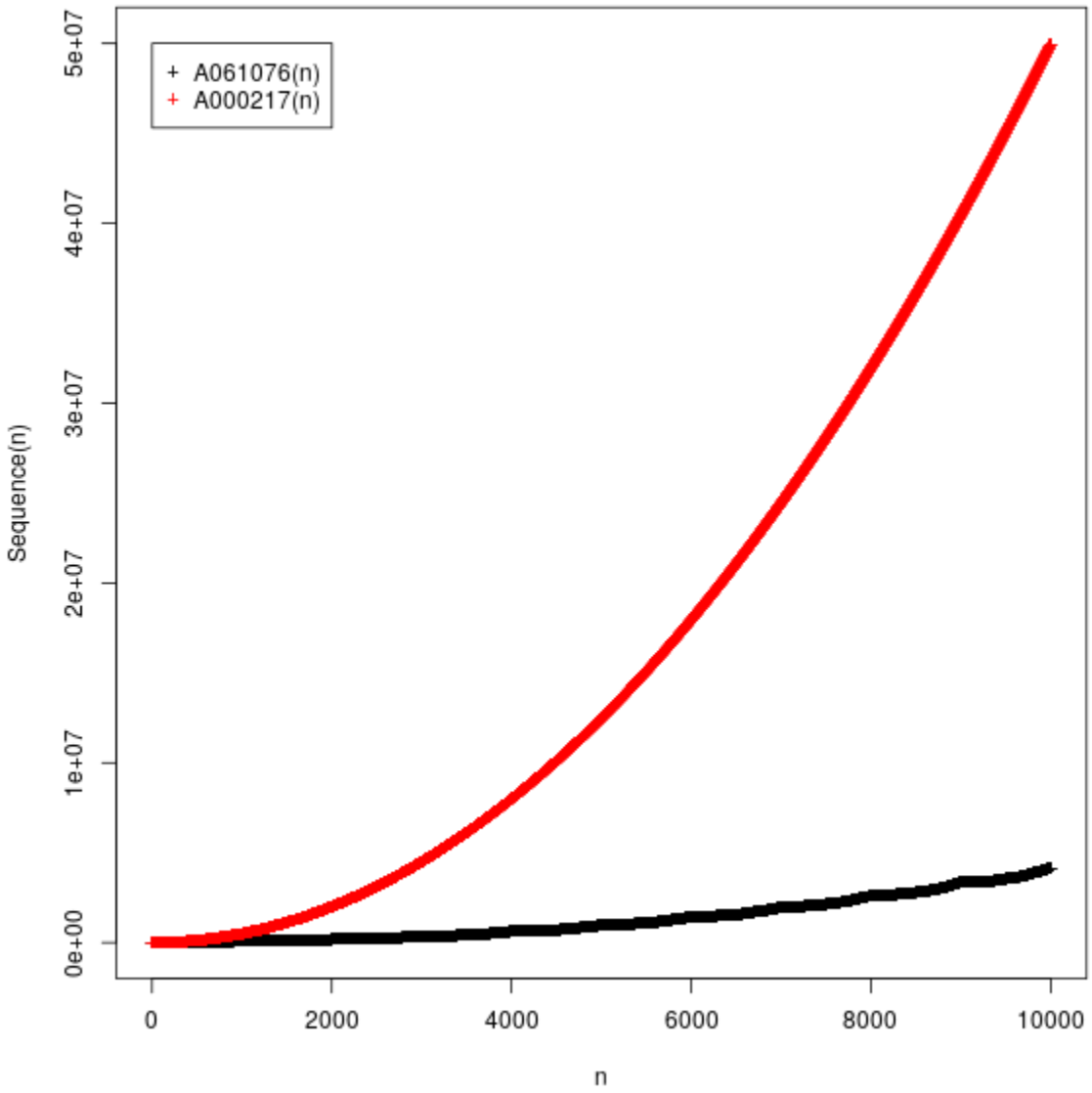
A apărut în secțiunea "Formula" începând cu 09.04 2022 și în secțiunea "Comments" din 10.04.2022, în contextul numărului A061076, respectiv comportamentul asimptotic într-un caz particular $n = 10^{k-1}$

Let b(1), b(2), ..., b(k) be the digits of the base-10 expansion of n: n = b(1)*10^(k-1) + b(2)*10^(k-2) + ... + b(k). Then a(n) = b(1)*b(2)*...*b(k) + (45^k-45)/44 + (1/2)*Sum_{i=1..k} b(1)*b(2)*...*b(i)*(b(i)-1)*45^(k-i).
- Mihai Teodor, Apr 09 2022

For n = 10^(k-1), the closed-form formula from Mihai Teodor (see Formula section) gives a(n) = (45^k - 45)/44, so lim_{n->oo} log(a(n))/log_10(n) = log(45) = 3.80666248977.... - Jon E. Schoenfield Apr 10 2022

Mai jos prezint comportamentul asimptotic al numerelor $s(n)$,

comparativ cu $\frac{n(n+1)}{2}$ *, care este* **A000217** .



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Program ce determină S(n) doar pentru un număr dat *n* (verificarea relației găsite de profesorul Mihai Teodor).

```
#include<iostream>
#include<iomanip>
#include <math.h>
using namespace std ;
int n,i,c,k=1,b[100000],a[100000];float p,s;
int main()
{
   cout<<"n=";cin>>n;
   while(n)
   {c=n%10;b[k]=c;
      k++;n=n/10;
   } k=k-1;cout<<k<<endl;
   for (i=1;i<=k;i++)
   {
      a[i]=b[k-i+1];cout<<"a["<<i<<"]="<<a[i]<<endl;
   }
   p=1;s=0;

   for (i=1;i<=k-1;i++)
```

```
    {s=s+(p*a[i]*(a[i]-1)/2+1)*pow(45,k-i);
     p=p*a[i];
    }
     s=s+p*a[k]*(a[k]+1)/2;
    cout << fixed << setprecision(10) << s; //cout<<s;
    return 0;
}
```

Program ce determină toate sumele până la un număr dat *n*:

```
#include<iostream>
using namespace std ;
int main()

{ int n,m,k,s=0,p[100000];
  cout<<"n=";cin>>n;
   for (k=1;k<=n;k++)
   {m=k;
     p[k]=1;
     while (m!=0)
      {p[k]=p[k]*(m%10);
       m=m/10;
```

```
    }
    s=s+p[k];
    cout<<"s("<<k<<")="<<s<<"      "<<endl;
  }
  return 0;
}
```

Soluția este prezentată sub forma mai multor pagini.

Link:

https://oeis.org › A061076